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Skyrmions and Faddeev-Hopf solitons

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This paper describes a natural one-parameter family of generalized Skyrme systems, which includes the usual SU(2) Skyrme model and the Skyrme-Faddeev system. Ordinary Skyrmions resemble polyhedral shells, whereas the Hopf-type solutions of the Skyrme-Faddeev model look like closed loops, possibly linked or knotted. By looking at the minimal-energy solutions in various topological classes, and for various values of the parameter, we see how the polyhedral Skyrmions deform into looplike Hopf Skyrmions.

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I. INTRODUCTION

Recent years have seen extensive progress on understanding the nature and dynamics of topological solitons [1], and, in particular, of Skyrmions. For the SU(2) Skyrme system, minimal-energy Skyrmions resemble polyhedral shells [2]; for example, the 3-Skyrmion looks like a tetrahedron [3]. On the other hand, the Hopf-type solitons in the Skyrme-Faddeev system (where the field takes values in the 2-sphere S^2) resemble closed loops, which may be linked or knotted [4]; for example, the 3-soliton in this system looks like a slightly twisted circular loop. This paper describes a natural one-parameter family of generalized Skyrme systems, which interpolates between the standard SU(2) Skyrme model and the Skyrme-Faddeev model. Its minimum-energy solutions interpolate between polyhedral Skyrmions and stringlike Hopf solitons.

The simplest way to describe the family is as follows. In the SU(2) Skyrme model, the field takes values in the 3-sphere S^3, with its standard metric. This 3-sphere is fibered over S^2 (the Hopf fibration); and instead of the standard metric on S^3, we can use a metric for which distances along the (one-dimensional) fibers are scaled by a factor which we denote $1-\alpha$. So $\alpha = 0$ gives the standard Skyrme system, whereas $\alpha = 1$ corresponds to the target space being the quotient S^2, namely, the Skyrme-Faddeev system. The global symmetry SO(4) in the $\alpha = 0$ case is broken to U(2) when $\alpha > 0$; and this in turn means that the generalized Skyrmion solutions for $\alpha > 0$ have less symmetry than those for $\alpha = 0$.

The system can also be formulated in terms of a pair $Z = (Z^1, Z^2)^t$ of complex scalars, and as such is related to condensed-matter systems in which there are two flavors of Cooper pairs [5]. The parameter α then appears, in particular, as the coefficient of a term $J_\mu J^\mu$, where $J_\mu = iZ^\dagger \partial_\mu Z$ is the current density.

In two spatial dimensions, and without the fourth-order Skyrme terms, the case $\alpha = 1$ corresponds to the CP^1 model. The generalization of this to $\alpha < 1$ was investigated in [6]. It arises as a modification of the CP^1 model which takes account of the effect of fermions (starting with a system which has fermions as well as bosons, and integrating out the fermionic degrees of freedom). In this case, there are explicit finite-energy static solutions (parametrized by α) which, for $\alpha = 1$, are the usual instanton solutions of the two-dimensional CP^1 model. In the three-dimensional case which is discussed below, one needs a Skyrme term to stabilize the solutions, and the solutions have to be obtained numerically.

II. FAMILY OF SKYRME SYSTEMS

Let us consider, first, the general situation of a map Φ from a 3-space (with local coordinates x^j and metric g_{jk}) to another 3-space (with local coordinates φ^a and metric H_{ab}). The Skyrme energy density $\mathcal{E}$ of such a map may be defined as follows [7], in terms of the differential $\partial_j \varphi^a$ of Φ. Define a 3×3 matrix D by

$$D_a^b = g^{jk}(\partial_j \varphi^c) H_{ac} (\partial_k \varphi^b). \tag{1}$$

Then $\mathcal{E} = \mathcal{E}_2 + \mathcal{E}_4$, where

$$\mathcal{E}_2 = \lambda_2 \mathrm{tr}(D), \qquad \mathcal{E}_4 = \tfrac{1}{2}\lambda_4[(\mathrm{tr}D)^2 - \mathrm{tr}(D^2)]. \tag{2}$$

Here λ_2 and λ_4 are constants. If the metric H_{ab} admits a group of symmetries (isometries), then these will correspond to (global) symmetries of the system.

In what follows, we take the target space to be the 3-sphere S^3 equipped with a one-parameter family of U(2)-invariant metrics. A particular member of this family is the standard SO(4)-invariant metric, and the corresponding system is the usual SU(2) Skyrme model. The family of metrics may be described as follows.

Let $Z = (Z^1, Z^2)^t$ denote a complex 2-vector satisfying the constraint $Z^\dagger Z = |Z^1|^2 + |Z^2|^2 = 1$ (where $Z^\dagger$ is the complex-conjugate row vector corresponding to the column vector Z). The set of all such vectors Z forms a 3-sphere. Note that the map $(Z^1, Z^2) \mapsto Z^1/Z^2$ is the standard Hopf fibration from S^3 to S^2, with Z^1/Z^2 being the usual stereographic coordinate on S^2. The standard

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metric G on S^3 corresponds to

$$ds^2 = dZ^\dagger dZ. \qquad (3)$$

Let ξ be the vector field obtained from the 1-form $\omega = -iZ^\dagger dZ$ by raising its index with the metric (3). This vector field ξ has unit length and is tangent to the fibers of the Hopf fibration. Our family of metrics, parametrized by the real number α, is taken to be $H = G - \alpha\omega \otimes \omega$. An alternative way to write H is

$$ds^2 = dZ^\dagger dZ + \alpha(Z^\dagger dZ)(Z^\dagger dZ). \qquad (4)$$

Note that both G and ω, and hence also H, are manifestly invariant under the U(2) transformations $Z \mapsto \Lambda Z$, where $\Lambda \in \mathrm{U}(2)$.

For $\alpha < 1$, the metric (4) is positive definite. But when $\alpha = 1$, it becomes degenerate, with ξ being a zero eigenvector; distances along the Hopf fibers are then zero, and the metric is, in effect, the standard metric on the quotient space $CP^1 \cong S^2$. In other words, our one-parameter family includes the standard 3-sphere ($\alpha = 0$) and the standard 2-sphere ($\alpha = 1$). We will restrict to the range $\alpha \leq 1$ for which the metric is non-negative; in fact, our interest is in the range $0 \leq \alpha \leq 1$, which interpolates between the Skyrme and the Skyrme-Faddeev systems.

The Lagrangian $\mathcal{L}$ of the generalized Skyrme system [consistent with the expressions (2) for the static energy density] may be described as follows. The vector Z determines an SU(2) matrix according to

$$U = \begin{bmatrix} Z^1 & -\bar{Z}^2 \\ Z^2 & \bar{Z}^1 \end{bmatrix}.$$

Write

$$U^\dagger \partial_\mu U = L_\mu = iL_\mu^a \sigma_a,$$
$$[L_\mu, L_\nu] = K_{\mu\nu} = iK_{\mu\nu}^a \sigma_a,$$

where the partial derivative is with respect to space-time coordinates x^μ, and where σ_a denotes the Pauli matrices. Then $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4$, where

$$\mathcal{L}_2 = \lambda_2 g^{\mu\nu}(L_\mu^a L_\nu^a - \alpha L_\mu^3 L_\nu^3), \qquad (5)$$

$$\mathcal{L}_4 = \tfrac{1}{8}\lambda_4 g^{\mu\nu} g^{\beta\gamma}[(1-\alpha)K_{\mu\beta}^a K_{\nu\gamma}^a + \alpha K_{\mu\beta}^3 K_{\nu\gamma}^3]. \qquad (6)$$

In this form, the global U(2) symmetry corresponds to $U \mapsto \Omega U \Gamma$, where Γ is an SU(2) matrix and $\Omega = \exp(i\theta\sigma_3)$ is a diagonal SU(2) matrix; note that this transformation preserves both $L_\mu^a L_\nu^a$ and L_μ^3.

If $\alpha = 0$, then $\mathcal{L}$ is the standard Skyrme Lagrangian. If $\alpha = 1$, on the other hand, we get the Skyrme-Faddeev system [4,8–14]. One way of seeing this is to replace the field Z by the unit 3-vector field $\vec{\psi} = Z^\dagger \vec{\sigma} Z$. Then $\mathcal{L}$ with $\alpha = 1$ becomes

$$\mathcal{L} = \frac{1}{4}\lambda_2(\partial_\mu \vec{\psi})^2 + \frac{1}{32}\lambda_4(\Omega_{\mu\nu})^2,$$

where $\Omega_{\mu\nu} = \vec{\psi} \cdot (\partial_\mu \vec{\psi}) \times (\partial_\nu \vec{\psi})$; this is the Skyrme-Faddeev Lagrangian.

If we take the space on which the field U is defined to be $\mathbf{R}^3$, then we need a boundary condition $U \to U_0$ (constant) as $r \to \infty$, to have finite energy. Fields satisfying this condition are classified topologically by their winding number $N = \int \mathcal{B} d^3x$, where $\mathcal{B}$ is the topological charge density

$$\mathcal{B} = \epsilon_{jkl}\mathrm{tr}(L_j L_k L_l)/(24\pi^2). \qquad (7)$$

In the limit $\alpha \to 1$, N equals the Hopf number of the S^2-valued field.

The values of the constants λ_2 and λ_4 correspond to the energy and length scales. To choose convenient values for them in what follows, let us consider the system defined on the unit 3-sphere S^3 (that is, take g_{jk} to be the standard metric on S^3) [7,12]; and take the field $Z(x^j)$ to correspond to the identity map from S^3 to itself (in other words, an isometry if $\alpha = 0$). It is straightforward to compute the energy E of this field: one gets

$$E = 2\pi^2(3-\alpha)\lambda_2 + 2\pi^2(3-2\alpha)\lambda_4.$$

So from now on let us take

$$\lambda_2 = 1/[4\pi^2(3-\alpha)], \qquad \lambda_4 = 1/[4\pi^2(3-2\alpha)].$$

Consequently, the "identity" field has unit energy for all $\alpha \in [0, 1]$.

III. FAMILIES OF SKYRMION SOLUTIONS

A numerical minimization procedure was used to find local minima of the static energy E for various values of N and α, and hence stable Skyrmion solutions; the results are described below. The procedure uses a finite-difference version of the functional E on a cubic grid, with a second-order scheme in which the truncation error is of order h^4 where h is the lattice spacing, and using the coordinate $1/x$ for $|x| > q \approx 1$ (similarly for y and z) so that the whole of $\mathbf{R}^3$ is included. With a relatively small number of lattice points (say 33^3), this achieves an accuracy of better than 1%. The discrete energy was then minimized using a standard conjugate-gradient method (flowing down the energy gradient). This produces a local minimum of the energy functional. In general, there are many local minima; the starting configuration determines which one is produced by this procedure. It seems likely that the solutions described below are global minima in the relevant topological classes, but the only evidence for this at present is consistency with previous studies in the $\alpha = 0$ and $\alpha = 1$ cases [2–4].

Most straightforward are the $N = 1$ and $N = 2$ cases, where the solutions admit a continuous symmetry. For $N = 1$, the $\alpha = 0$ Skyrmion has O(3) (spherical) symmetry, and energy $E = 1.232$. When $\alpha > 0$, this is broken to O(2) (axial) symmetry. The normalized energy $E(\alpha)$ depends smoothly on α, and the numerical results indicate that, to within the small numerical error, its dependence is quadratic: $E(\alpha) = 1.232 - 0.008\alpha^2$. The topological charge density (7) has an almost-spherical shape, for all α.

For $N = 2$, one has O(2) symmetry both for $\alpha = 0$ and $\alpha = 1$, and the constant-$\mathcal{B}$ surfaces resemble tori. So the expectation is that the $N = 2$ generalized Skyrmions will look like tori for all values of α, with $E(\alpha)$ decreasing from $E(0) = 2.358$ to $E(1) = 2.00$ [10,13] over the range $\alpha \in [0, 1]$, but this has not been checked.

It is worth remarking at this point on the energy values of Skyrme-Faddeev solitons given in [4], so as to facilitate comparison with that paper. The energies in [4] should be divided by a factor of $32\pi^2\sqrt{2}$ in order to adjust the normalization to the one being used here, and by a further factor of (about) 0.93 to allow for the fact that [4] used a finite-size box (rather than all of $\mathbf{R}^3$). For example, in the $N = 2$ case, [4] gives an energy $E_{\mathrm{BS}} = 835$, which

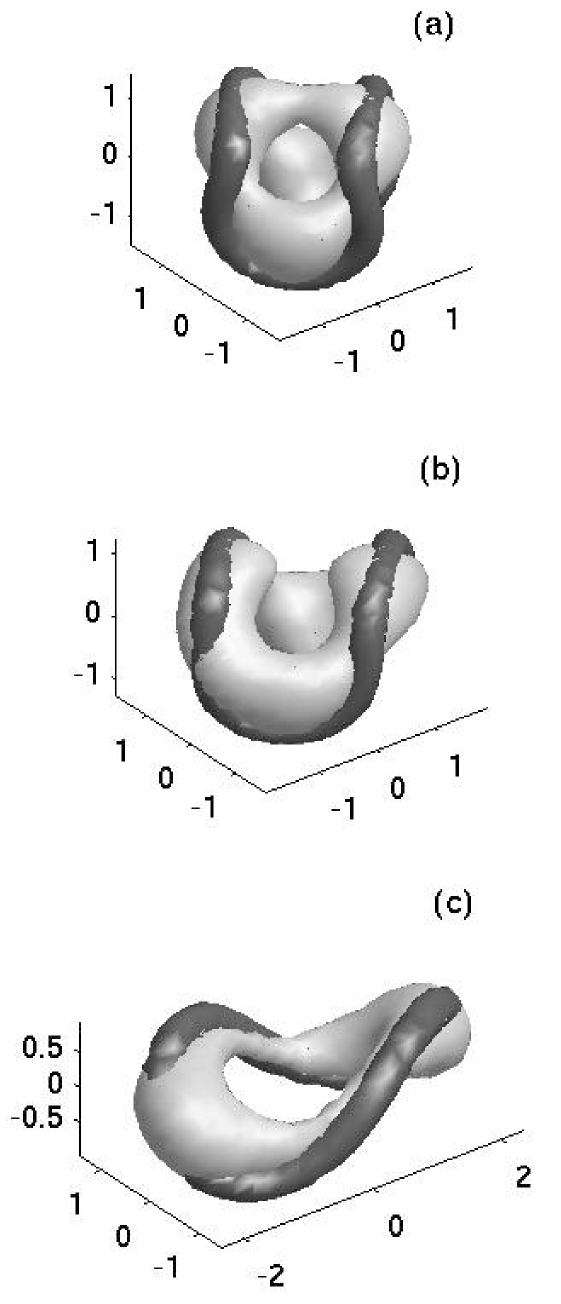

FIG. 1. The charge density isosurface and position curve $\psi_3 = -0.8$ of the $N = 3$ generalized Skyrmions: for (a) $\alpha = 0$, (b) $\alpha = 0.2$, and (c) $\alpha = 0.4$.

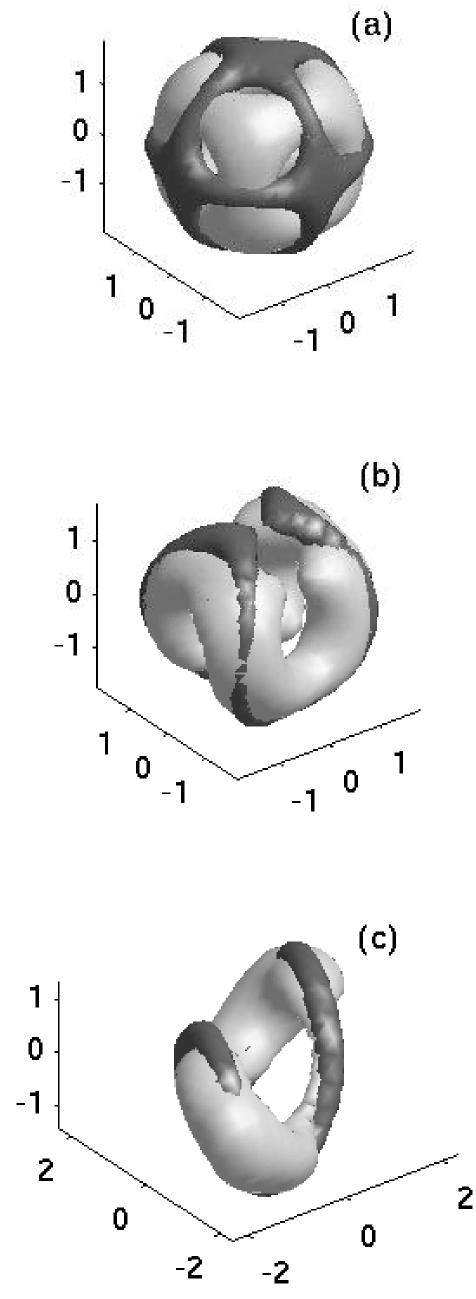

FIG. 2. The charge density isosurface and position curve $\psi_3 = -0.8$ of the $N = 4$ generalized Skyrmions: for (a) $\alpha = 0$, (b) $\alpha = 0.35$, and (c) $\alpha = 0.4$.

when divided by the two factors above yields $E = 2.01$. This is within 0.5% of the correct figure.

For $N \geq 3$, the picture is less straightforward, with the Skyrmions having at most discrete symmetry. We look in detail at the cases $N = 3$ and $N = 4$. The 3-Skyrmion (for $\alpha = 0$) has energy $E = 3.4386$ and tetrahedral symmetry [2,3]; in particular, a typical constant-$\mathcal{B}$ surface resembles a tetrahedron. It is also useful to plot the curve in $\mathbf{R}^3$ where $\psi_3 = -1$, or equivalently where $Z^1 = 0$ and $|Z^2| = 1$; in the Faddeev-Skyrme system, this curve may be interpreted as the position of the stringlike Hopf Skyrmion [4]. Each plot in Fig. 1 depicts the surface $\mathcal{B}(\mathbf{x}) = (\max\mathcal{B})/2$, with the "thickened" curve $\psi_3 = -0.8$ strung around it; 1(a) is for $\alpha = 0$, 1(b) is for $\alpha = 0.2$, and 1(c) is for $\alpha = 0.4$. We see that as α increases from zero, the tetrahedral Skyrmion transforms into a twisted torus or loop (see also the pictures in [4] for the $\alpha = 1$ case). The tetrahedral symmetry is broken to the subgroup D_2. The normalized energy $E(\alpha)$ again has a quadratic dependence on α: $E(\alpha) \approx 3.4386 - 0.60\alpha^2$.

Finally, let us look at the case $N = 4$. The 4-Skyrmion (for $\alpha = 0$) resembles a cube [2,3]: see Fig. 2(a), where the same quantities are plotted as in Fig. 1. As α increases, the minimum-energy configuration becomes a closed loop strung along eight edges of the cube [Fig. 2(b), for $\alpha = 0.35$], which then flattens as α increases further. When $\alpha = 1$, one again gets a twisted circular loop, with the twisting being greater than in the $N = 3$ case (see also the pictures in [4]).

We have seen that the Skyrme model and the Skyrme-Faddeev-Hopf system may be regarded as members of a one-parameter family of generalized Skyrme systems, and the topological-soliton solutions of all these systems, although rather different in appearance, are all closely related to one another. A recent paper [15] has pointed out a similarity between sphaleron solutions of the Skyrme system and axially symmetric Hopf solitons, especially as the winding number N increases. These solutions are unstable (saddle points of their respective energy functionals), and this connection between Skyrmions and Hopf solitons is quite different from the one described above. It may be of interest, however, to investigate sphaleron-type solutions of the family of Skyrme systems and see how they depend on the family parameter α.

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Note added.—A similar family arises from considering bundles of strings [S. Nasir and A. J. Niemi, Mod. Phys. Lett. A **17**, 1445 (2002)]. The author is grateful to Professor Niemi for correspondence regarding this.

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